NAME: ________________________

TEACHER: ________________________

The Scots College

Year 12 Mathematics Extension 2

Assessment 1

February 2011

General Instructions

- Working time - 45 minutes
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question

TOTAL MARKS: 36

WEIGHTING: 10 %

Question	Topic	Max Marks	Marks Obtained
1	Complex Numbers	18	
2	Graphs	18	
Total		**36**	

Question 1 (18 Marks)

a) (i) Express $\frac{1+i}{1-i}$ in modulus-argument form. [2]

(ii) Hence, or otherwise, find the value of $\left(\frac{1+i}{1-i}\right)^8$. [1]

b) Find the square roots of $(-7+24i)$ in the form $a+ib$, where $a, b \in R$. [3]

c) Let 0, z_1 and z_2 be complex numbers represented in the Argand diagram by O, P and Q respectively. If the point $\frac{z_1}{z_2} \neq 0$, lies on the imaginary axis, show that $OP \perp OQ$. [2]

d) (i) Indicate the region in the Argand diagram that satisfies simultaneously [3]
$|z-(1+i)| \leq 1$ and $\frac{\pi}{4} \leq \arg([z-(1+i)] \leq \frac{\pi}{2}$.

(ii) Find the maximum value of $|z|$ in the region mentioned in (i) and also find the specific complex number z that corresponds to this maximum value. [2]

e) Let a complex number be given by $z = \cos\theta + i\sin\theta$.

(i) Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$. [1]

(ii) By factorizing z^6+1 , express z^6+1 as a product of three **quadratic** factors of z. [2]

(iii) By using the results of (i) and (ii), or otherwise, show that [2]
$\cos 3\theta = \cos\theta\left(2\cos\theta - \sqrt{3}\right)\left(2\cos\theta + \sqrt{3}\right)$

………..Question 2 on next page

Question 2 (18 Marks)

a) Consider the function $y = \frac{1+x^2}{x}$.

(i) Find the stationary points and points of inflexion, if any, for the function. [2]

(ii) Find the equation(s) of any asymptote(s). [1]

(iii) Sketch the graph of $y = f(x)$, showing clearly any intercepts, stationary points and points of inflexion, if they exist, and equation of any asymptote. [3]

(iv) Hence, sketch, on the same set of axis as in (iii), the graph of $y = \frac{x}{1+x^2}$, indicating clearly the intercepts, stationary points and asymptotes. [2]

b) Consider the curve $2x^2 + xy - y^2 = 0$. Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point (2,4) on the curve. [4]

c) The graph given below (see next page) represents the function $y = 1 - 2e^{-x^2}$. The line $y = 1$ is the horizontal asymptote. In the graphs provided in the answer booklet, sketch the function represented by

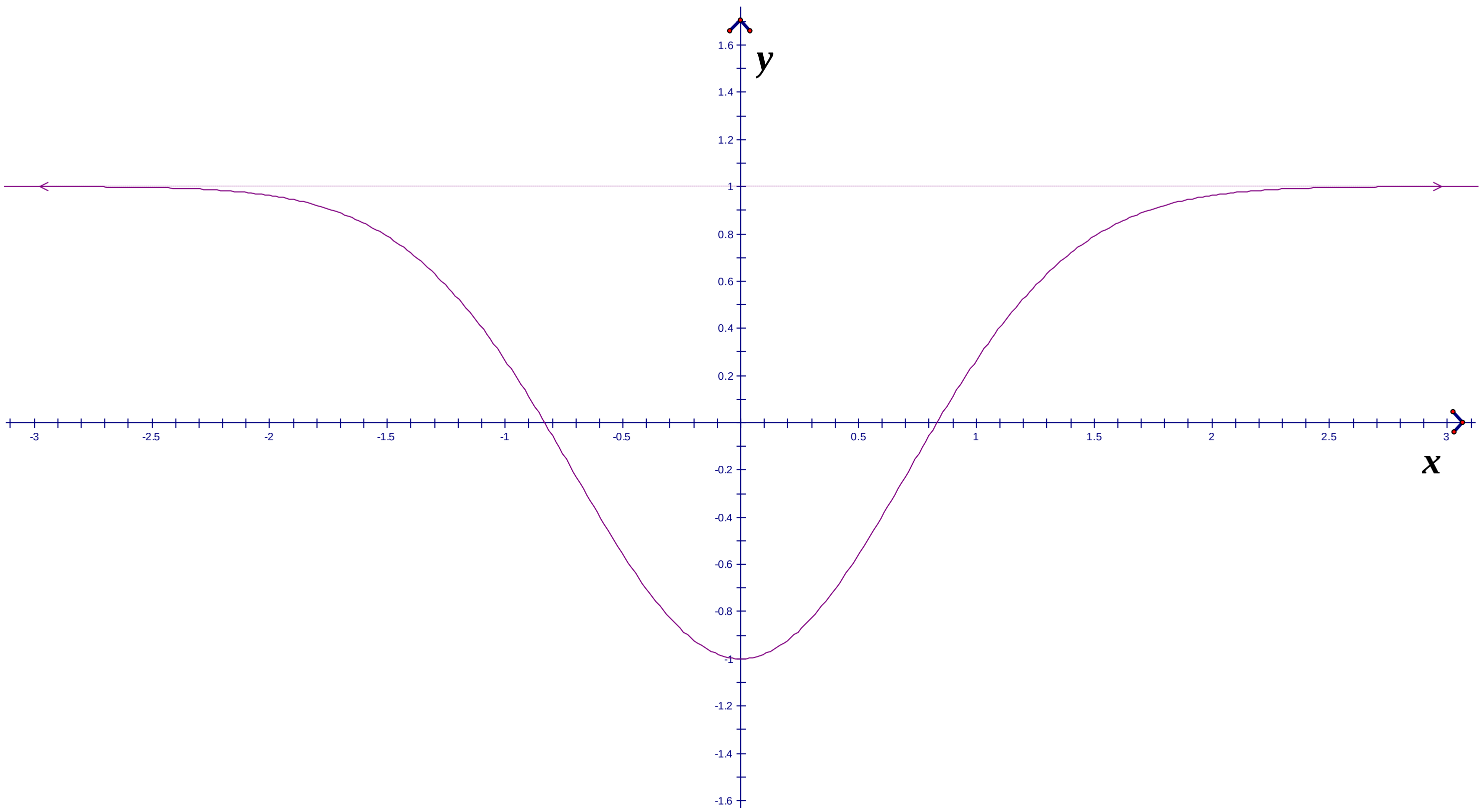
(i) $y = [f(x)]^2$ [2]

(ii) $y^2 = f(x)$ [2]

(iii) $|y| = f(x)$ [2]

……..**Question 2 continued on next page**

Question 2 c) : Graph of $y = 1 - 2e^{-x^2}$.



... END OF EXAMINATION …

Mathematics Extension 2. Assessment 1.

ANSWER SHEET

Feb 2011

Name: SOLUTIONS

Teacher: ______

Question No. 1.

(a) (i) $\dfrac{1+i}{1-i}$

$= \dfrac{\sqrt{2}\ \text{cis}\ \pi/4}{\sqrt{2}\ \text{cis}\ (-\pi/4)}$ ✓

$= \text{cis}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)$

$= \text{cis}\ \pi/2$

$= \underline{\underline{\cos \pi/2 + i \sin \pi/2}}$ ✓

(ii) $\text{cis}\ \pi/2 = i$

$\therefore \dfrac{1+i}{1-i} = i$

$\left(\dfrac{1+i}{1-i}\right)^8 = i^8 = \underline{\underline{1}}$ ✓

(b) let $\sqrt{-7+24i} = a+ib$

$-7+24i = a^2+2iab-b^2$

$a^2-b^2 = -7$ — ①

$2ab = 24 \Rightarrow ab = 12$ — (ii) $\Rightarrow b = 12/a$ ✓

$a^2 - \dfrac{144}{a^2} = -7 \Rightarrow a^4+7a^2-144 = 0$

$(a^2+16)(a^2-9) = 0$

Q 1 (b)

$(a^2+16)(a^2-9) = 0$

$\therefore a^2 = -16$ (rej $\because a \in R$) or $a^2 = 9$

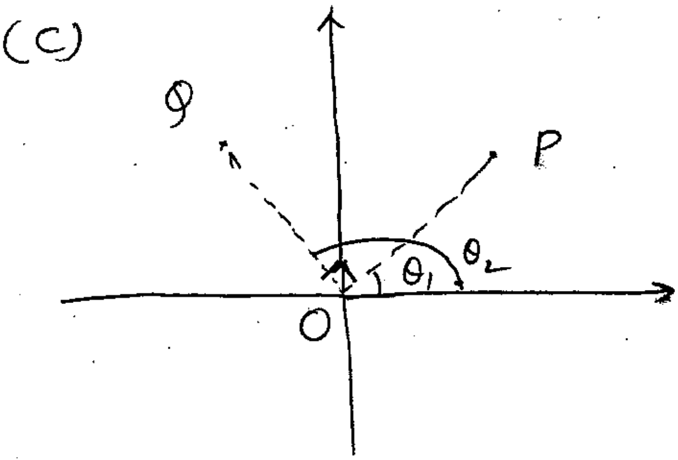
$a = \pm 3$

when $a = 3$, $b = 4$

$a = -3$ $b = -4$

$\therefore$ square roots of $-7+24i$ are $\underline{\underline{\pm(3+4i)}}$

(c)



if $\frac{z_2}{z_1}$ lies on the imaginary axis then

$\frac{z_2}{z_1} = ki$ where k is a real no.

$\therefore z_2 = k i z_1$

$= k \operatorname{cis}\frac{\pi}{2} \cdot z_1$

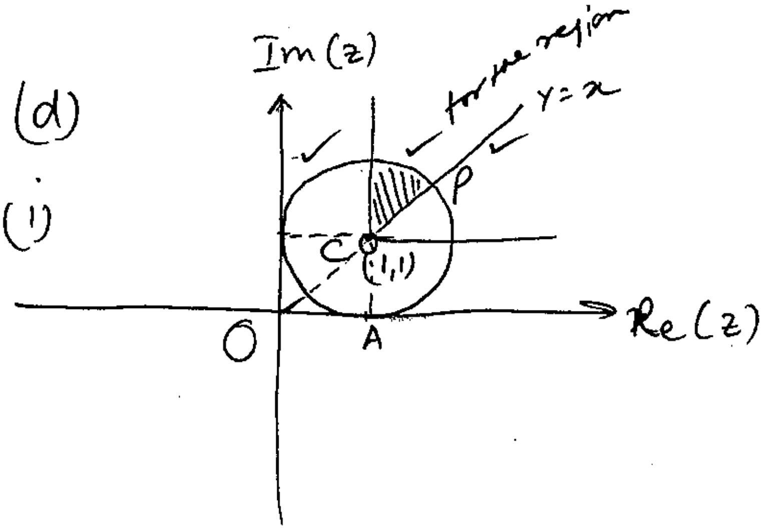
$\arg(z_2) = \arg z_1 + \frac{\pi}{2}$

$\therefore \angle POQ = 90°$

or $\underline{\underline{OP \perp OQ}}$.

Q 1 (d)

(i)



(ii) The maximum value of $|z| = OP$ which corresponds to the complex no. representing P (lying on the line $y = x$).

$CP = 1 \quad OC = \sqrt{OA^2 + AC^2}$

$= \sqrt{1+1} = \sqrt{2}$

$\therefore \ OP = 1 + \sqrt{2}$

$\therefore$ Max. value of $|z| = 1 + \sqrt{2}$

P is representing $z = (1+\sqrt{2}) \operatorname{cis} \pi/4$

or $z = \dfrac{1+\sqrt{2}}{\sqrt{2}} + i \, \dfrac{1+\sqrt{2}}{\sqrt{2}}$

$z = \dfrac{\sqrt{2}+2}{2} + \dfrac{\sqrt{2}+2}{2} i$

(e) (i) $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta, \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$

(ii) $z^6+1 = (z^2)^3 + 1^3$

$= (z^2+1)(z^4 - z^2 + 1)$

$= (z^2+1)((z^2+1)^2 - 3z^2)$

$= (z^2+1)(z^2+1-\sqrt{3}z)(z^2+1+\sqrt{3}z)$

(iii) $z^6+1 = (z^2+1)(z^2+1-\sqrt{3}z)(z^2+1+\sqrt{3}z) \div z^3.$

$z^3 + \frac{1}{z^3} = \left(z+\frac{1}{z}\right)\left(z+\frac{1}{z} - \sqrt{3}\right)\left(z+\frac{1}{z}+\sqrt{3}\right)$

or $2\cos 3\theta = 2\cos\theta(2\cos\theta - \sqrt{3})(2\cos\theta + \sqrt{3})$

(from the results of (i))

$\cos 3\theta = \cos\theta(2\cos\theta - \sqrt{3})(2\cos\theta + \sqrt{3}).$

ANSWER SHEET

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Question No. 2 .

(a) $y = \frac{1+x^2}{x}$

$= \frac{1}{x} + x$

(i) $\frac{dy}{dx} = -\frac{1}{x^2} + 1$ $\qquad \frac{d^2y}{dx^2} = \frac{2}{x^3}$

for st. pt:

$\frac{dy}{dx} = 0 \qquad -\frac{1}{x^2} + 1 = 0$

$\frac{1}{x^2} = 1$

$x^2 = 1$

$x = 1, \ y = \frac{1+1}{1} = 2$

or $x = -1, \ y = \frac{1+1}{-1} = -2$

when $x = 1, \ \frac{d^2y}{dx^2} = \frac{2}{1} > 0 \ \therefore$ $\underline{(1,2) \text{ is a min. pt}}$

$x = -1, \ \frac{d^2y}{dx^2} = \frac{2}{-1} < 0 \ \therefore$ $\underline{(-1,-2) \text{ is a max. pt}}$

for poi $\frac{d^2y}{dx^2} = 0$ or $\frac{2}{x^3} = 0$

no soln. $\therefore$ no poi

Q2 (a)

(ii) $y = \frac{1}{x} + x$

when $x \to \infty$, $\frac{1}{x} \to 0$, $y \to x$

$\therefore$ $y = x$ is the oblique asymptote

$x = 0$ is the vertical asymptote

(iii) $f(x) = \frac{1+x^2}{x}$,

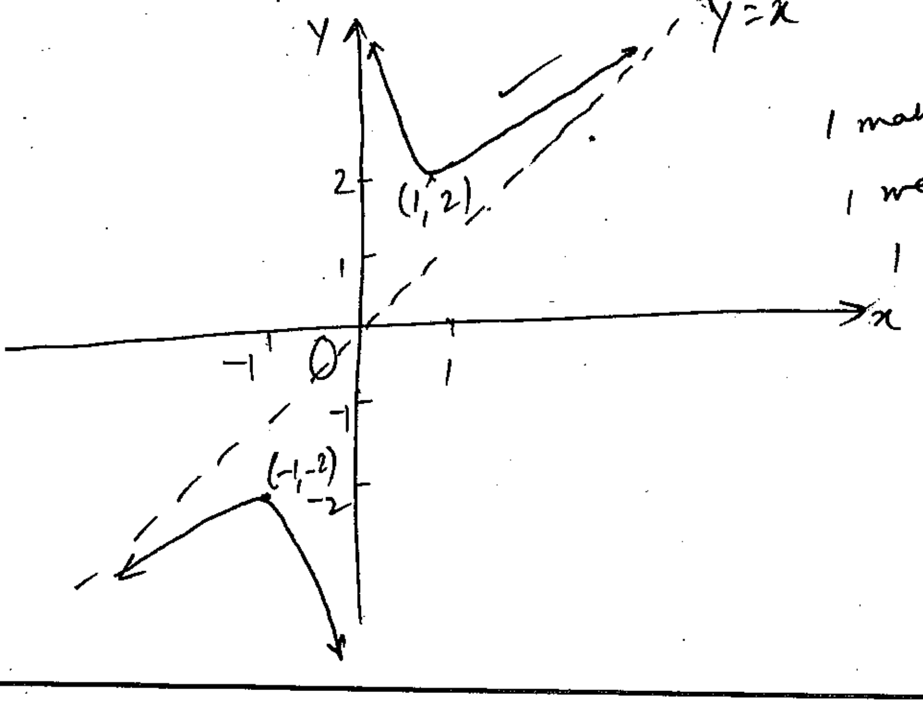
$x = 0$ y is undefined $\therefore$ no y-int

$y = 0$
$1 + x^2 = 0$
no solution
$\therefore$ no x-int.

$f(-x) = \frac{1+(-x)^2}{-x} = -\frac{1+x^2}{x}$

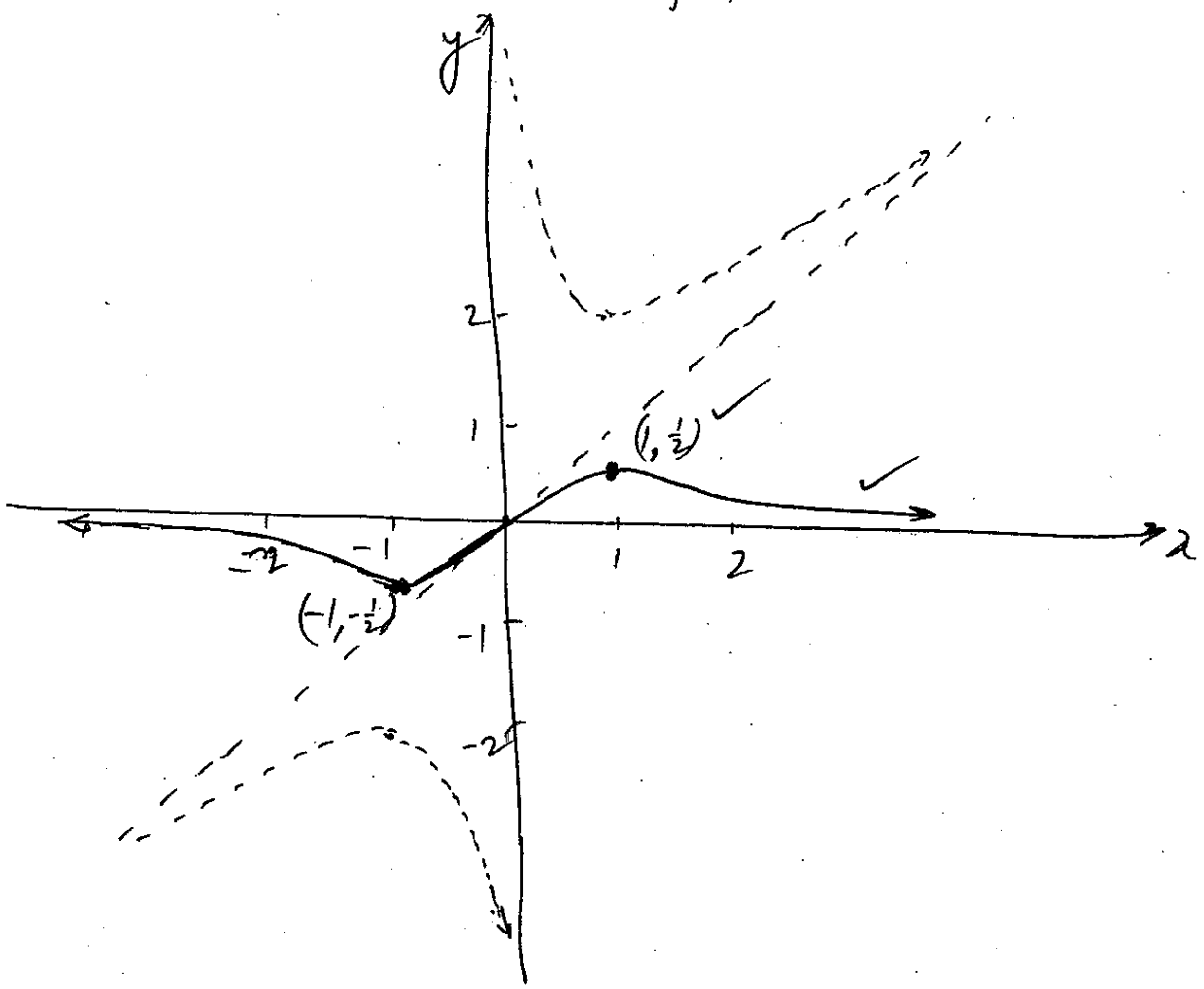
$f(-x) = -f(x)$

$\therefore$ $y = \frac{1+x^2}{x}$ is an odd function



1 mark st. pt. & asymptote
1 mark shape
1 mark labels

Q 2(a) (iv) $y = \frac{x}{1+x^2} = \frac{1}{f(x)}$



Q 2 (b)

$$2x^2 + xy - y^2 = 0$$

Differentiating w.r.t. x

$$4x + x\frac{dy}{dx} + y(1) - 2y\frac{dy}{dx} = 0$$

$$4x + y = (2y - x)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{4x+y}{2y-x}. \checkmark \qquad v = 2y - x \qquad u = 4x + y$$

$$v' = 2y' - 1 \qquad u' = 4 + y'$$

$$\frac{d^2y}{dx^2} = \frac{(2y-x)(4+y') - (4x+y)(2y'-1)}{(2y-x)^2} \checkmark$$

When $x = 2,\ y = 4$

$$\frac{dy}{dx} = \frac{8+4}{8-2} = \frac{12}{6} = \underline{\underline{2}} \checkmark$$

$$\frac{d^2y}{dx^2} = \frac{(8-2)(4+2) - (8+4)(4-1)}{(8-2)^2}$$

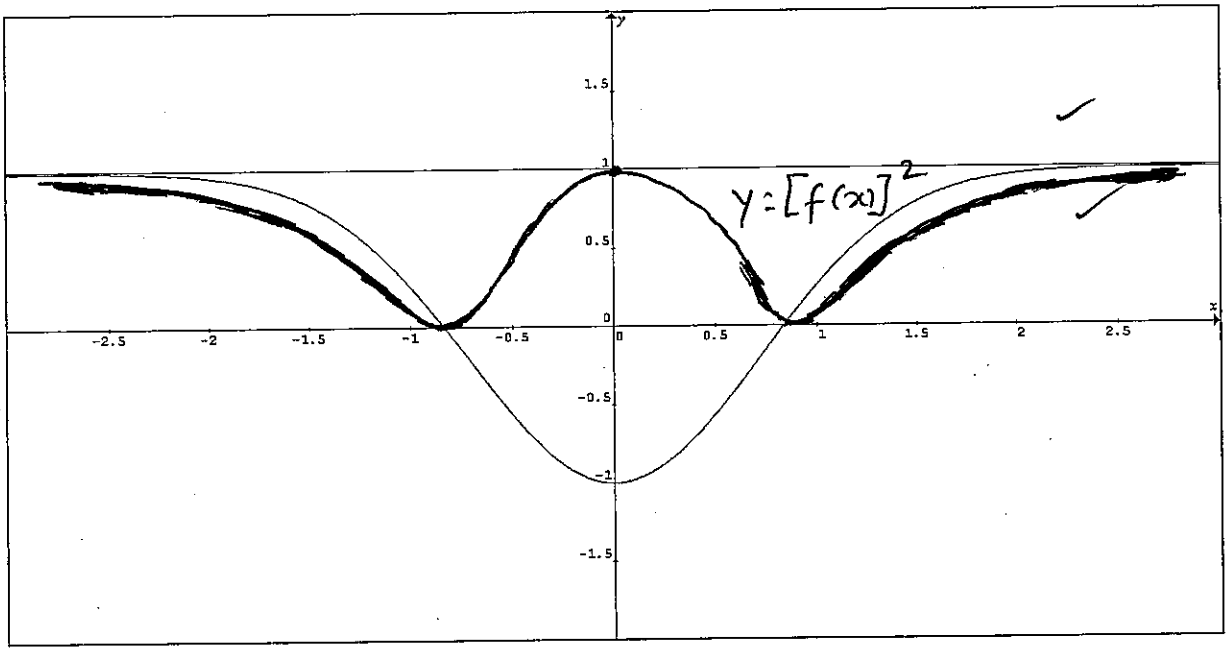
$$= \frac{36 - 36}{36} = -\frac{0}{36}$$

$$= \underline{\underline{0}} \checkmark$$

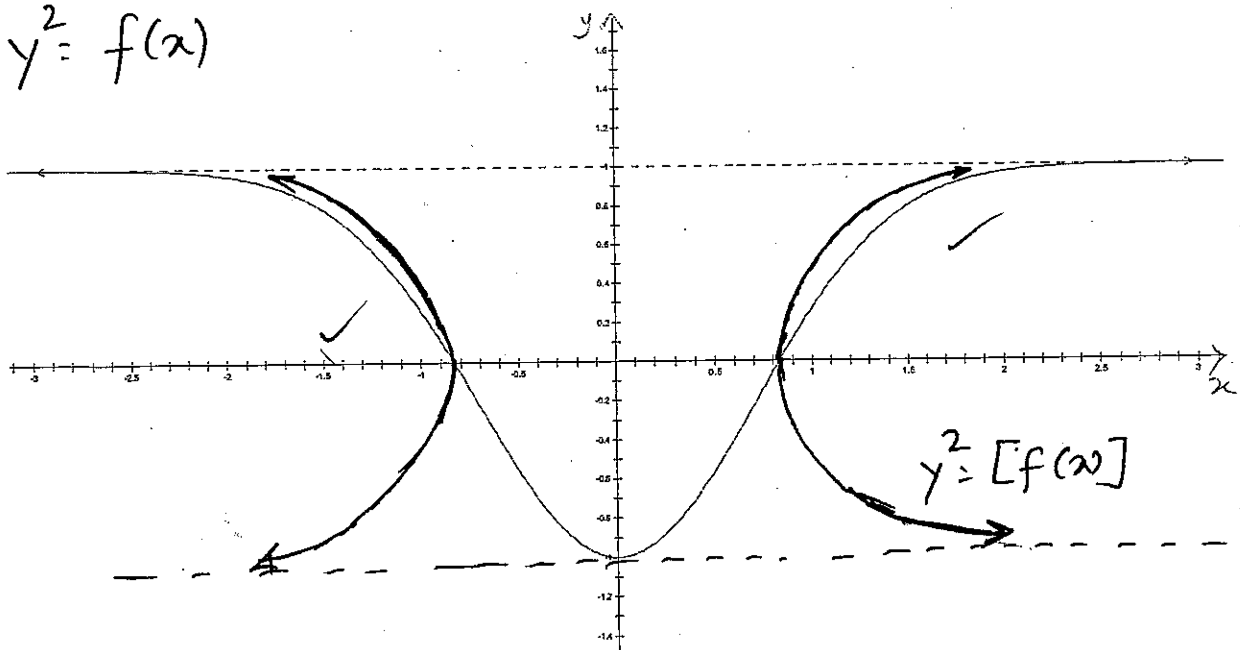
Q2 (a)

-1 for any mistake in each graph.

(i) $y = [f(x)]^2$



(ii) $y^2 = f(x)$



Q 2(c)

(iii) $|y| = f(x)$